# Optimization techniques for energy systems 

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June 30, 2021

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## Why study optimization?

Optimization is about making good decisions or choices in a rigorous way, often subject to constraints. Applications appear everywhere in science, mathematics, and business.

Examples:

- portfolio optimization
- variables: amounts invested in different assets
- constraints: budget, max/min investment per asset, minimum return
- objective: overall risk or return variance
- data fitting
- variables: model parameters
- constraints: prior information, parameter limits
- objective: measure of misfit or prediction error
- energy usage
- variables: turning OFF/ON, state of charge, power generation
- constraints: operational limits, timing requirements, power balance
- objective: power consumption/bill, pollution


## Optimization problem types

We generally consider families or classes of optimization problems, characterized by particular forms of the objective and constraint functions.

The optimization problem is called a linear program (LP) if the objective function and the constraint are linear.

- Integer Programming Problem (IP): all variables are restricted to be integer
- 0-1 Integer Programming Problem (BIP): all variables are restricted to be binary
- Mixed Integer Linear Programming (MILP): some of the variables are restricted to be integers.


## Optimization problem

A common problem format:

$$
\begin{array}{rl}
\min _{z \in Z} & f(z) \\
\text { subject to: } & g_{i}(z) \leq 0, \quad i=1, \ldots, m \\
& h_{j}(z)=0 \quad j=1, \ldots, p
\end{array}
$$

- Objective function $f: Z \rightarrow \mathbb{R}$;
- Domain $Z \subseteq \mathbb{R}^{n}$ of the objective function, from which the decision variables $z:=\left\{z_{1}, z_{2}, \ldots, z_{n}\right\}$ must be chosen;
- Optional inequality constraint function $g_{i}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ for $i=1, \ldots, m$;
- Optional equality constraint function $h_{j}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ for $j=1, \ldots, p$;
- The $g_{i}$ and $h_{j}$ define the constraint set $S \subseteq Z$.


## Properties of the optimization problem

Consider the generic optimization problem:

$$
J^{*}=\min _{z \in S} f(z)
$$

Notation:

- if $J^{*}=-\infty$, unbounded below problem;
- if $S$ is empty, infeasible problem $\left(J^{*}=+\infty\right)$;
- if $S=\mathbb{R}^{n}$, unconstrained problem;
- there might be more than one solution: $\operatorname{argmin} f(z)=\left\{z \in S \mid f(z)=J^{*}\right\}$ $z \in S$
- Feasible point: a vector $z \in S$ satisfying the inequality and equality constraints, i.e. $g_{i}(z) \leq 0$ for $i=1, \ldots, m, h_{j}(z)=0$ for $j=1, \ldots, p$.
- Strictly feasible point: a vector $z \in Z$ satisfying the inequality constraints strictly, i.e. $g_{i}(z)<0$ for $i=1, \ldots, m$.


## Terminology

$$
z^{*}=\arg \min _{z} f(z)
$$

- Optimal value: The lowest possible objective value, $f\left(z^{*}\right)$.
- Optimal solution or minimizer: Any feasible $z^{*} \in Z$ such that $f\left(z^{*}\right) \leq f(z)$ for all feasible $z \in Z$.
- Local optimum: a point $z_{\text {local }}^{*}$ that is optimal within a neighbourhood $\left\|z-z_{\text {local }}^{*}\right\| \leq R$.


## Active, Inactive and Redundant Constraints

Consider the standard problem

$$
\begin{array}{rl}
\min _{z \in Z} & f(z) \\
\text { subject to: } & g_{i}(z) \leq 0, \quad i=1, \ldots, m \\
& h_{j}(z)=0 \quad j=1, \ldots, p
\end{array}
$$

- The $i^{\text {th }}$ inequality constraint $g_{i}(z) \leq 0$ is active at $\bar{z}$ if $g_{i}(\bar{z})=0$. Otherwise it is inactive;
- Equality constraints are always active;
- A redundant constraint is one that does not change the feasible set. This implies that removing a redundant constraint does not change the solution. Example:

$$
\begin{array}{rl}
\min _{z \in \mathbb{R}} & f(z) \\
\text { subject to: } & z \leq 1 \\
& z \leq 2 \quad \text { (redundant) }
\end{array}
$$

## Geometry of an Optimization Problem



## "Easier" problems: Linear and Convex Quadratic Programs

Linear Program (LP): Linear cost and constraint functions; feasible set is a polyhedron.

$$
\min _{z} c^{\top} z
$$

subject to: $\quad G z \leq h$

$$
A z=0
$$

Convex Quadratic Program (QP): Quadratic cost and linear constraint functions; feasible set is a polyhedron. Convex if $P \succeq 0$.

$$
\min _{z} \frac{1}{2} z^{\top} P z+q^{\top} z
$$


subject to: $\quad G z \leq h$

$$
A z=0
$$

## "Harder" problems: Nonconvex and Integer Programs

Nonconvex Quadratic Program:
QP with $P \nsucceq 0$.

$$
\begin{aligned}
\min _{z} & \frac{1}{2} z^{\top} P z+q^{\top} z \\
\text { subject to: } & G z \leq h \\
& A z=0
\end{aligned}
$$

Mixed Integer Linear Program
(MILP): Linear program with binary or integer constraints.

$$
\begin{aligned}
\min _{z} & c^{\top} z \\
\text { subject to: } & G z \leq h \\
& A z=0 \\
& z \in\{0,1\}^{n} \text { or } z \in \mathbb{Z}^{n}
\end{aligned}
$$



## Convex set

- A set $Z$ is convex if and only if for any pair of points $z$ and $y$ in $Z$, any convex combination of $z$ and $y$ lies in $Z$.

$$
\begin{aligned}
& 1 \text { of } z \text { and } y \text { lies in } Z \text {. } \\
& Z \text { is convex } \Leftrightarrow \lambda z+(1-\lambda) y=Z, \forall \lambda \in[0,1], \forall z, y \in Z
\end{aligned}
$$

- All line segments starting and ending in $Z$ stay within $Z$.


## Convex:



Non-convex:


## Convex set

- An affine set is a convex set defined by $Z=\left\{z \in \mathbb{R}^{n} \mid A z=b\right\}$. A subspace is an affine set with $b=0$.
- A hyperplane is defined by $Z=\left\{z \in \mathbb{R}^{n} \mid a^{\top} z=b\right\}$ for $a \neq 0$, where $a \in \mathbb{R}^{n}$ is the normal vector to the hyperplane.
- A halfspace is everything on one side of a hyperplane, i.e. $\left\{z \in \mathbb{R}^{n} \mid a^{\top} z \leq b\right\}$ for $a \neq 0$. It can either be open (strict inequality) or closed (non-strict inequality).
- Hyperplanes and halfspaces are always convex.


An affine space in $\mathbb{R}^{2}$


A hyperplane


A closed halfspace

## Convex set

- A polyhedron is the intersection of a finite number of closed halfspaces:

$$
Z=\left\{z \mid a_{1}^{T} z \leq b_{1}, a_{2}^{T} z \leq b_{2}, \ldots, a_{m}^{T} z \leq b_{m}\right\}=\{z \mid A z \leq b\}
$$

- A polytope is a bounded polyhedron.
- Polyhedra and polytopes are always convex.


An (unbounded) polyhedron


A polytope

## Convex function

- A function $f: \operatorname{dom}(f) \rightarrow \mathbb{R}$ is convex if and only if its domain $\operatorname{dom}(f)$ is convex and

$$
f(\lambda x+(1-\lambda) y) \leq \lambda f(x)+(1-\lambda) f(y), \forall \lambda \in[0,1], \forall z, y \in \operatorname{dom}(f)
$$

- The function f is strictly convex if the above inequality is strict for $\lambda \in(0,1)$.



## Convex function

## 1st-order condition for convexity

- A differentiable function $f: \operatorname{dom}(f) \rightarrow \mathbb{R}$ with a convex domain is convex iff

$$
f(y) \geq f(z)+\nabla f(z)^{\top}(y-z), \forall z, y \in \operatorname{dom}(f)
$$

i.e., a first order approximator of $f$ around any point $z$ is a global underestimator of $f$.

- The gradient $\nabla f(z)$ is given by $\nabla f(z)=\left[\frac{\partial f(z)}{\partial z_{1}}, \frac{\partial f(z)}{\partial z_{2}}, \ldots, \frac{\partial f(z)}{\partial z_{n}}\right]$



## Convex function

## 2nd-order condition for convexity

- A twice-differentiable function $f: \operatorname{dom}(f) \rightarrow \mathbb{R}$ is convex iff its domain $\operatorname{dom}(f)$ is convex and

$$
\nabla^{2} f(z) \succeq 0, \quad \forall z \in \operatorname{dom}(f)
$$

where the Hessian square matrix $\nabla^{2} f(z)$ is defined by $\nabla^{2} f(z)_{i j}=\frac{\partial^{2} f(z)}{\partial z_{i} \partial z_{j}}$.

- If $\operatorname{dom}(f)$ is convex and $\nabla^{2} f(z) \succ 0$ for all $z \in \operatorname{dom}(f)$, then $f$ is strictly convex.


## Convex function

## Epigraph of a function

- The epigraph of a function $f$ is the set

$$
\operatorname{epi}(f)=\left\{\left.\left[\begin{array}{l}
z \\
t
\end{array}\right] \right\rvert\, z \in \operatorname{dom}(f), f(z) \leq t\right\} \subseteq \operatorname{dom}(f) \times \mathbb{R}
$$



- $f$ is convex iff epi $(f)$ is convex.


## Convex function

## Level and sublevel sets

- The level set $L_{\alpha}$ of a function $f$ for value $\alpha$ is the set of all $z \in \operatorname{dom}(f)$ for which $f(z)=\alpha$, i.e.

$$
L_{\alpha}=\{z \mid z \in \operatorname{dom}(f), f(z)=\alpha\}
$$

For $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ these are contour lines of constant "height".

- The sublevel set $C_{\alpha}$ of a function $f$ for value $\alpha$ is defined by

$$
C_{\alpha}=\{z \mid z \in \operatorname{dom}(f), f(z) \leq \alpha\}
$$

Function $f$ is convex $\Rightarrow$ sublevel sets of $f$ are convex for all $\alpha$. But $\nLeftarrow$ !

## Convex function

Examples of convex functions: $\mathbb{R} \rightarrow \mathbb{R}$

- The following functions are convex (on domain $\mathbb{R}$ unless otherwise stated):
- Affine: $a x+b$ for any $a, b \in \mathbb{R}$
- Exponential: $e^{a x}$ for any $a \in \mathbb{R}$

- Powers: $z^{\alpha}$ on domain $\mathbb{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- Powers of absolute value: $|z|^{p}$, for $p \geq 1$
- The following functions are concave (i.e., their opposite is convex) on domain $\mathbb{R}$ unless otherwise stated:
- Affine: $a x+b$ for any $a, b \in \mathbb{R}$
- Powers: $z^{\alpha}$ on domain $\mathbb{R}_{++}$, for $0 \leq \alpha \leq 1$
- Logarithm: $\log z$ on domain $\mathbb{R}_{++}$
- Entropy: $-z \log z$ on domain $\mathbb{R}_{++}$



## Convex function

Examples of convex functions: $\mathbb{R}^{n} \rightarrow \mathbb{R}$

- Affine functions on $\mathbb{R}^{n}$ are both convex and concave:
- On $\mathbb{R}^{n}$, for some $a \in \mathbb{R}^{n}$ and $b \in \mathbb{R}$

$$
f(z)=a^{\top} z+b
$$

- Vector Norms on $\mathbb{R}^{n}$ are all convex:
- On $\mathbb{R}^{n}, l_{p}$ norms have the form, for $p \geq 1$,



## Convex optimization problem

## Standard form of the convex optimization problem



- the objective function $f$ is a convex function;
- equality constraints are affine;
- inequality constraints functions $g_{i}$ are convex.


## Theorem

For a convex optimization problem, every locally optimal solution is globally optimal

## General Linear Problem

Affine cost and constraint functions

$$
\begin{aligned}
\min _{z} & c^{\top} z+d \\
\text { subject to: } & G z \leq h \\
& A z=b
\end{aligned}
$$



- Feasible set is a polyhedron.
- Constant component $d$ can be left out - it has no effect on the optimal solution.
- Many problems can be written (with some effort) into LPs.
- Huge variety of solution methods and software are available.


## General Linear Problem

## Example: Cheapest cat-food problem:

- Choose quantities $z_{1}, z_{2}, \ldots, z_{n}$ of $n$ different foods with unit cost $c_{j}$.
- Each food $j$ has nutritional content $a_{i j}$ for nutrient $i$.
- Require for each nutrient $i$ a minimum level $b_{i}$.

In linear program form:

$$
\begin{array}{rll}
\min _{z} & c^{\top} z & \\
\text { subject to: } & A z \geq b & b-\notin z \leq 0 \\
& z \geq 0 & -z \leq 0
\end{array}
$$

This is an example of a resource allocation problem.

## General Quadratic Problem

Quadratic cost function with $P \succeq 0$, affine constraint functions:

$$
\min _{z} \quad \frac{1}{2} z^{\top} P z+q^{\top} z+r
$$

subject to: $\quad G z \leq h$

$$
A z=b
$$



- Feasible set is a polyhedron.
- Constant component $r$ can be left out - it has no effect on the optimal solution.


## General Quadratic Problem

$$
A z=b
$$

## Example: Least Squares



- Analytical solution $A^{\dagger} b\left(A^{\dagger}\right.$ is the pseudo-inverse).
- Extra linear constraints $l \leq z \leq u$ can be added, although the QP would no longer have an analytical solution.

In case the problem above is ill-posed, it can be regularized as

$$
\min _{z}\|A z-b\|+r(z)
$$

The particular case $r(z)=\rho\|z\|_{1}$ yields to a LASSO problem (Least Absolute Shrinkage and Selection Operator).

## General Quadratic Problem

## Example: Linear program with random cost

$$
\begin{array}{rl}
\min _{z} & \mathbb{E}\left[c^{\top} z\right]+\gamma \operatorname{var}\left(c^{\top} z\right)=\bar{c}^{\top} z+\gamma z^{\top} \Gamma z \\
\text { subject to: } & G z \leq h \\
& A z=b
\end{array}
$$

- Random cost function vector $c$ with mean $\bar{c}$ and covariance $\Gamma$, we penalize expected cost plus a risk premium on the variance.
- Large $\gamma$ means large risk aversion, we prefer a small variance to the lowest expected cost.


## Classification problems

## Classification via Logistic Regression

We are given $N$ measurement points $p_{i}$ and an associated label $l_{i} \in\{-1,1\}$. We look for a separating hyperplane in the $p$ space, $w^{T} p+b=0$, whose parameters $w$ and $b$ can be determined as solution to the following (strictly) convex unconstrained problem

$$
\min _{w, b} \sum_{i=1}^{N} \ln [1+e^{-\underset{\underbrace{\left.w^{T} p_{i}+b\right)}_{-1} l_{i}}{+1}]}+\frac{\rho}{2}\|w\|^{2}
$$

## Classification problems

Classification via Support Vector Machines, hard margins

$$
\begin{array}{ll}
\min _{w, b} & \frac{1}{2}\|w\|^{2} \\
\text { s.t. } & \left(w^{T} p_{i}+b\right) l_{i} \geq 1, \quad i \in\{1, \ldots, N\}
\end{array}
$$

Classification via Support Vector Machines, soft margins

$$
\begin{aligned}
& \min _{w, b} \frac{1}{2}\|w\|^{2}+\rho \sum_{i} \xi_{i} \\
& \text { s.t. }\left(w^{T} p_{i}+b\right) l_{i} \geq 1-\xi_{i}, \quad i \in\{1, \ldots, N\} \\
& \xi \geq 0
\end{aligned}
$$

## Unconstrained problems

## Optimality Criterion for Differentiable $f$ 's

## Theorem (Necessary condition)

$f: \mathbb{R}^{s} \rightarrow \mathbb{R}$ is differentiable at $\bar{z}$. If $\bar{z}$ is a local minimizer, then $\nabla f(\bar{z})=0$.

## Theorem (Sufficient condition)

Suppose that $f: \mathbb{R}^{s} \rightarrow \mathbb{R}$ is twice differentiable at $\bar{z}$. If $\nabla f(\bar{z})=0$ and the Hessian of $f(z)$ at $\bar{z}$ is positive definite, then $\bar{z}$ is a local minimizer.

## Theorem (Necessary and sufficient condition)

Suppose that $f: \mathbb{R}^{s} \rightarrow \mathbb{R}$ is differentiable at $\bar{z}$. If $f$ is convex, then $\bar{z}$ is a local minimizer, if and only if $\nabla f(\bar{z})=0$.

## Constrained problems

## Optimality Conditions

Consider the problem

$$
\begin{array}{rl}
\min _{z \in Z} & f(z) \\
\text { subject to: } & g_{i}(z) \leq 0, \quad i=1, \ldots, m \\
& h_{j}(z)=0 \quad j=1, \ldots, p
\end{array}
$$

- In general, an analytical solution does not exist.
- Solutions are usually computed by recursive algorithms which start from an initial guess $z_{0}$ and at step $k$ generate a point $z_{k}$ such that $\left\{f\left(z_{k}\right)\right\}_{k=0,1, \ldots}$ converges to $f^{*}$.
- These algorithms recursively use and/or solve analytical conditions for optimality.


## Constrained problems

tromush - Kuhn - Tucker

## KKT optimality conditions

$z^{*},\left(\lambda^{*}, \nu^{*}\right)$ of an optimization problem, with differentiable cost and constraints and zero duality gap, have to satisfy the following conditions:

$$
\begin{align*}
& 0=\nabla f\left(z^{*}\right)+\sum_{i=1}^{m} \lambda_{i}^{*} \nabla g_{i}\left(z^{*}\right)+\sum_{j=1}^{p} \nu_{j}^{*} \nabla h_{i}\left(z^{*}\right)  \tag{ia}\\
& 0=\lambda_{i}^{*} g_{i}\left(z^{*}\right), i=1, \ldots, m  \tag{ib}\\
& 0 \leq \lambda_{i}^{*}, i=1, \ldots, m  \tag{ic}\\
& 0 \geq g_{i}\left(z^{*}\right), i=1, \ldots, m  \tag{id}\\
& 0=h_{j}\left(z^{*}\right), j=1, \ldots, p \tag{le}
\end{align*}
$$

Conditions (1a)-(1e) are called the Karush-Kuhn-Tucker (KKT) conditions.

## The Lagrange Function

To the primal optimization problem

$$
\begin{array}{rll}
\inf _{z} & f(z) & \\
\text { subject to: } & g_{i}(z) \leq 0 \quad \text { for } i=1, \ldots, m \\
& h_{i}(z)=0 \quad \text { for } i=1, \ldots, p \\
& z \in Z &
\end{array}
$$

we associate the Lagrange function

$$
L(z, \lambda, \nu)=f(z)+\lambda^{\top} g(z)+\nu^{\top} h(z)
$$

- $\lambda_{i} \geq 0$ and $\nu_{i}$ called Lagrange multipliers or dual variables
- the objective is augmented with weighted sum of constraint functions
- notice $f(z)+\lambda^{\top} g(z)+\nu^{\top} h(z) \leq f(z)$ for feasible $z$

We look for $\min _{z} L(z, \lambda, \nu)$ without constraints.
Then (dual problem) we maximize w.r.t. $(\lambda, \nu)$.

## The Lagrange Function

To the primal optimization problem

$$
\begin{array}{rll}
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## The Lagrange Function

To the primal optimization problem

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\begin{array}{rll}
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\text { subject to: } & g_{i}(z) \leq 0 \quad \text { for } i=1, \ldots, m \\
& h_{i}(z)=0 \quad \text { for } i=1, \ldots, p \\
& z \in Z &
\end{array}
$$

we associate the Lagrange function

$$
\begin{aligned}
& \text { nge function } \leq 0 \leq: ~=0 \\
& L(z, \lambda, \nu)=f(z)+\lambda^{\top} g(z)+\nu^{\top} h(z)
\end{aligned}
$$

- $\lambda_{i} \geq 0$ and $\nu_{i}$ called Lagrange multipliers or dual variables
- the objective is augmented with weighted sum of constraint functions
- notice $f(z)+\lambda^{\top} g(z)+\nu^{\top} h(z) \leq f(z)$ for feasible $z$

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## The Lagrange Function

To the primal optimization problem

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\begin{array}{cll}
\inf _{z} & f(z) & \\
\text { subject to: } & g_{i}(z) \leq 0 \quad \text { for } i=1, \ldots, m \\
& h_{i}(z)=0 \quad \text { for } i=1, \ldots, p \\
& z \in Z &
\end{array}
$$

we associate the Lagrange function

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L(z, \lambda, \nu)=f(z)+\lambda^{\top} g(z)+\nu^{\top} h(z)
$$

- $\lambda_{i} \geq 0$ and $\nu_{i}$ called Lagrange multipliers or dual variables
- the objective is augmented with weighted sum of constraint functions
- notice $f(z)+\lambda^{\top} g(z)+\nu^{\top} h(z) \leq f(z)$ for feasible $z$

We look for $\min _{z} L(z, \lambda, \nu)$ without constraints.
Then (dual problem) we maximize w.r.t. $(\lambda, \nu)$.

## Software tools for optimization I

$$
\begin{aligned}
\min _{z} & \left|z_{1}+5\right|+\left|z_{2}-3\right| \\
\text { subject to: } & 2.5 \leq z_{1} \leq 5 \\
& -1 \leq z_{2} \leq 1
\end{aligned}
$$

## - CVX toolbox for MATLAB

```
% Initialize CVX environment
cvx_begin
% Define cost function
variables z1 z2;
% Define constraints
minimize(abs(z1 + 5) + abs(z2 - 3));
subject to;
2.5<= z1<= 5;
-1<= z2<= 1;
cvx_end % solves automatically
```


## Software tools for optimization II

- YALMIP toolbox for MATLAB

```
% Initialize yalmip environment
yalmip('clear');
% Decision variables
sdpvar z1 z2;
const= [];
obj= 0;
% Define cost function
obj = obj+ abs(z1 + 5) + abs(z2 - 3);
% Define constraints
Const= [Const, 2.5<= z1<= 5];
Const= [Const, -1<= z2<= 1];
% Solve the optimization problem
solvesdp(Const,obj);
```


## Mixed integer linear programming (MILP) I

- A mixed-integer linear programming (MILP) is a problem of the form

where $x \in \mathbb{Z}^{q} \times \mathbb{R}^{p}$. and scalars $b_{1}, \ldots, b_{m} \in \mathbb{R}$ are problem parameters that specify the objective and the constraint functions.
- $q$ variables are integers, $p$ variables are continuous $(q+p=n)$.
- Feasible solution: The set $S$ of all $x \in \mathbb{Z}^{q} \times \mathbb{R}^{p}$ which satisfy the linear constraints $a_{i}^{T} x \leq b_{i}, \quad i=1, \ldots, m$

$$
S=\left\{x \in \mathbb{Z}^{q} \times \mathbb{R}^{p}, a_{i}^{T} x \leq b_{i}, i=1, \ldots, m\right\}
$$

is called feasible set and an element $x \in S$ is called feasible solution.

## Mixed integer linear programming (MILP) II

- Different approaches There is no single technique for solving integer programs. Instead, a number of procedures have been developed:
- Enumeration (Guaranteed to find a feasible solution, but exponential growth in computation time)
- Branch and Bound
- Cutting plane


## They require LP relaxation.

## LP-relaxation



- LP constraints form a polytope
- IP feasible set is given by set of all integer-valued points within the polytope
feasible set of IP $\subset$ feasible set of LP


## LP-relaxation

The LP-relaxation of a MILP or IP is obtained by removing the integer constraints on all variables.
e.g., in the binary case replace $x \in\{0,1\}$ by $0 \leq x \leq 1$.

## Branch and Bound I

The main idea of branch and bound consists in dividing a computationally hard problem into easier subproblems and systematically exploit the information gained from solving these subproblems.

Tree search where the tree is built using three main steps:

- Branch Pick a variable and divide the problem in two subproblems at this variable.
- Bound Solve the LP-relaxation to determine the best possible objective value for the node.
- Prune Prune the branch of the tree (i.e., the tree will not be developed any further in this node) if
- the subproblem is infeasible
- the best achievable objective value is worse than a known optimum


## Branch and Bound: example

$$
\begin{aligned}
& \max _{x} z=2 x_{1}+3 x_{2} \\
& \text { s.to }-1.3 x_{1}+3 x_{2} \leq 9 \\
& 3 x_{1}+0.9 x_{2} \leq 18 \\
& x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{aligned}
$$


$\max _{x} z=2 x_{1}+3 x_{2}$ s.to $-1.3 x_{1}+3 x_{2} \leq 9$ $\begin{aligned} \text { s.to }-1.3 x_{1}+3 x_{2} & \leq 9 \\ 3 x_{1}+0.9 x_{2} & \leq 18\end{aligned}$

$$
x_{1}, x_{2} \geq 0 \in \mathbb{Z}
$$



(S2)
$\max z=2 x_{1}+3 x_{2}$



## Branch and Bound: example

$$
\begin{aligned}
& \max _{x} z=2 x_{1}+3 x_{2} \\
& \text { s.to }-1.3 x_{1}+3 x_{2} \leq 9 \\
& 3 x_{1}+0.9 x_{2} \leq 18 \\
& x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{aligned}
$$

$\max _{x} z=2 x_{1}+3 x_{2}$
s.to $-1.3 x_{1}+3 x_{2} \leq 9$ $3 x_{1}+0.9 x_{2} \leq 18$
$\max z=2 x_{1}+3 x_{2}$
s.to $-1.3 x_{1}+3 x_{2} \leq 9$ $3 x_{1}+0.9 x_{2} \leq 18$

$$
x_{1}, x_{2} \geq 0 \in \mathbb{Z}
$$



$$
\begin{aligned}
z^{*} & =23.89 \\
x_{1}^{*} & =4.51 \\
x_{2}^{*} & =4.95
\end{aligned}
$$



$$
4 \leq 4.51 \leq 5
$$



## Branch and Bound: example

$$
\begin{aligned}
& \max _{x} z=2 x_{1}+3 x_{2} \\
& \text { s.to }-1.3 x_{1}+3 x_{2} \leq 9 \\
& 3 x_{1}+0.9 x_{2} \leq 18 \\
& x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{aligned}
$$



$$
\begin{aligned}
z^{*} & =23.89 \\
x_{1}^{*} & =4.51 \\
x_{2}^{*} & =4.95
\end{aligned}
$$

$$
\begin{array}{r}
(\mathrm{S} 1) \\
\max _{x} z=2 x_{1}+3 x_{2} \\
\text { s.to }-1.3 x_{1}+3 x_{2} \leq 9 \\
3 x_{1}+0.9 x_{2} \leq 18 \\
x_{1} \leq 4 \\
x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{array}
$$

## Branch and Bound: example

$$
\begin{align*}
& \text { (S1) }  \tag{S2}\\
& \max _{x} z=2 x_{1}+3 x_{2} \\
& \text { s.to }-1.3 x_{1}+3 x_{2} \leq 9 \\
& 3 x_{1}+0.9 x_{2} \leq 18 \\
& x_{1} \leq 4 \\
& x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{align*}
$$

$$
\begin{aligned}
& \max _{x} z=2 x_{1}+3 x_{2} \\
& \text { s.to }-1.3 x_{1}+3 x_{2} \leq 9 \\
& 3 x_{1}+0.9 x_{2} \leq 18
\end{aligned}
$$



$$
x_{1}, x_{2} \geq 0 \in \mathbb{Z}
$$




## Branch and Bound: example

## (S1)

$$
\begin{aligned}
& \max _{x} z=2 x_{1}+3 x_{2} \\
& \text { s.to }-1.3 x_{1}+3 x_{2} \leq 9 \\
& \qquad 3 x_{1}+0.9 x_{2} \leq 18
\end{aligned}
$$

$$
\begin{align*}
& \max _{x} z=2 x_{1}+3 x_{2}  \tag{S2}\\
& \text { s.to }-1.3 x_{1}+3 x_{2} \leq 9 \\
& 3 x_{1}+0.9 x_{2} \leq 18
\end{align*}
$$

$$
x_{1} \leq 4
$$

$$
x_{1} \geq 5
$$

$$
x_{1}, x_{2} \geq 0 \in \mathbb{Z}
$$

$$
x_{1}, x_{2} \geq 0 \in \mathbb{Z}
$$

(S1)

$$
\begin{aligned}
& z^{*}=22.2 \\
& x_{1}^{*}=4 \\
& x_{2}^{*}=4.73
\end{aligned}
$$

$$
4 \leq 4.73 \leq 5
$$

$$
\begin{align*}
z^{*} & =20  \tag{S2}\\
x_{1}^{*} & =5 \\
x_{2}^{*} & =3.3
\end{align*}
$$

## Branch and Bound: example

## (S1)

$$
\begin{aligned}
& \max _{x} z=2 x_{1}+3 x_{2} \\
& \text { s.to }-1.3 x_{1}+3 x_{2} \leq 9 \\
& 3 x_{1}+0.9 x_{2} \leq 18
\end{aligned}
$$

$$
x_{1} \leq 4
$$

$$
x_{1}, x_{2} \geq 0 \in \mathbb{Z}
$$

$$
\begin{align*}
& z^{*}=22.2  \tag{S1}\\
& x_{1}^{*}=4 \\
& x_{2}^{*}=4.73
\end{align*}
$$

$$
\begin{align*}
& \max _{x} z=2 x_{1}+3 x_{2}  \tag{S2}\\
& \text { s.to }-1.3 x_{1}+3 x_{2} \leq 9 \\
& 3 x_{1}+0.9 x_{2} \leq 18
\end{align*}
$$

$$
x_{1} \geq 5
$$

$$
x_{1}, x_{2} \geq 0 \in \mathbb{Z}
$$

$$
\begin{align*}
z^{*} & =20  \tag{S2}\\
x_{1}^{*} & =5 \\
x_{2}^{*} & =3.3
\end{align*}
$$

## Branch and Bound: example

$$
\begin{aligned}
\max _{x} z=2 x_{1}+3 x_{2} & \\
\text { s.to }-1.3 x_{1}+3 x_{2} & \leq 9 \\
3 x_{1}+0.9 x_{2} & \leq 18 \\
x_{1} & \leq 4 \\
x_{1}, x_{2} \geq 0 & \in \mathbb{Z}
\end{aligned}
$$

## (S11)

$$
\begin{aligned}
\max _{x} z=2 x_{1}+3 x_{2} & \\
\text { s.to }-1.3 x_{1}+3 x_{2} & \leq 9 \\
3 x_{1}+0.9 x_{2} & \leq 18 \\
x_{1} & \leq 4
\end{aligned}
$$

$$
\begin{gathered}
x_{2} \leq 4 \\
x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{gathered}
$$



## (S12)

$$
\begin{aligned}
\max _{x} z=2 x_{1}+3 x_{2} & \\
\text { s.to }-1.3 x_{1}+3 x_{2} & \leq 9 \\
3 x_{1}+0.9 x_{2} & \leq 18 \\
x_{1} & \leq 4 \\
x_{2} \geq 5 & \\
x_{1}, x_{2} \geq 0 & \in \mathbb{Z}
\end{aligned}
$$

## Branch and Bound: example

## (S11)

$$
\begin{aligned}
\max _{x} z=2 x_{1}+3 x_{2} & \\
\text { s.to }-1.3 x_{1}+3 x_{2} & \leq 9 \\
3 x_{1}+0.9 x_{2} & \leq 18 \\
x_{1} & \leq 4 \\
x_{2} \leq 4 & \\
x_{1}, x_{2} \geq 0 & \in \mathbb{Z}
\end{aligned}
$$

$$
\begin{aligned}
& z^{*}=20 \\
& x_{1}^{*}=4 \\
& x_{2}^{*}=4
\end{aligned}
$$

## (S12)

$$
\begin{aligned}
\max _{x} z=2 x_{1}+3 x_{2} & \\
\text { s.to }-1.3 x_{1}+3 x_{2} & \leq 9 \\
3 x_{1}+0.9 x_{2} & \leq 18 \\
x_{1} & \leq 4 \\
x_{2} \geq 5 & \\
x_{1}, x_{2} \geq 0 & \in \mathbb{Z}
\end{aligned}
$$



Infeasible subproblem

## Branch and Bound: example

## (S11)

$$
\begin{aligned}
& \max _{x} z=2 x_{1}+3 x_{2} \\
& \text { s.to }-1.3 x_{1}+3 x_{2} \leq 9 \\
& 3 x_{1}+0.9 x_{2} \leq 18 \\
& x_{1} \leq 4 \\
& x_{2} \leq 4 \\
& x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{aligned}
$$



$$
\begin{array}{|c|}
\hline z^{*}=20 \\
x_{1}^{*}=4 \\
x_{2}^{*}=4 \\
\hline
\end{array}
$$



## (S12)

$$
\begin{aligned}
\max _{x} z=2 x_{1}+3 x_{2} & \\
\text { s.to }-1.3 x_{1}+3 x_{2} & \leq 9 \\
3 x_{1}+0.9 x_{2} & \leq 18 \\
x_{1} & \leq 4 \\
x_{2} \geq 5 & \\
x_{1}, x_{2} \geq 0 & \in \mathbb{Z}
\end{aligned}
$$

(S12)
Infeasible subproblem
Pruned node

## Cutting plane

The main idea of the Cutting plane algorithm consists in an iterative reduction of the feasible region:

- solve LP-relaxation and obtain fractional solution
- add a new constraint (cut) that removes the fractional solution from the feasible set of the LP-relaxation



## Cutting plane: example

$$
\begin{aligned}
\max _{x} z= & 5 x_{1}+8 x_{2} \\
\text { s.to } x_{1}+x_{2} & \leq 6 \\
5 x_{1}+9 x_{2} & \leq 45 \\
& x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{aligned}
$$

## Cutting plane: example

$$
\begin{aligned}
& \max _{x} z= 5 x_{1}+8 x_{2} \\
& \text { s.to } x_{1}+x_{2} \leq 6 \\
& 5 x_{1}+9 x_{2} \leq 45 \\
& x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{aligned}
$$



## Cutting plane: example

$$
\begin{aligned}
& \max _{x} z= 5 x_{1}+8 x_{2} \\
& \text { s.to } x_{1}+x_{2} \leq 6 \\
& \leq x_{1}+9 x_{2} \leq 45 \\
& x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{aligned}
$$




## Cutting plane: example

$$
\begin{aligned}
\max _{x} z= & 5 x_{1}+8 x_{2} \\
\text { s.to } x_{1}+x_{2} & \leq 6 \\
5 x_{1}+9 x_{2} & \leq 45 \\
& x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{aligned}
$$




## Cutting plane: example

$$
\begin{aligned}
\max _{x} z=5 x_{1}+8 x_{2} & \\
\text { s.to } x_{1}+x_{2} & \leq 6 \\
5 x_{1}+9 x_{2} & \leq 45 \\
x_{1}, x_{2} \geq 0 & \in \mathbb{Z}
\end{aligned}
$$



$$
\begin{aligned}
& x_{1}^{*}=2.25 \\
& x_{2}^{*}=3.75
\end{aligned}
$$



## Cutting plane: example

$$
\begin{aligned}
\max _{x} z= & 5 x_{1}+8 x_{2} \\
\text { s.to } x_{1}+x_{2} & \leq 6 \\
5 x_{1}+9 x_{2} & \leq 45 \\
x_{1}, x_{2} \geq 0 & \in \mathbb{Z}
\end{aligned}
$$



$$
\begin{aligned}
& x_{1}^{*}=2.25 \\
& x_{2}^{*}=3.75
\end{aligned}
$$



## Cutting plane: example

$$
\begin{aligned}
\max _{x} z=5 x_{1}+8 x_{2} & \\
\text { s.to } x_{1}+x_{2} & \leq 6 \\
5 x_{1}+9 x_{2} & \leq 45 \\
x_{1}, x_{2} \geq 0 & \in \mathbb{Z}
\end{aligned}
$$



$$
\begin{aligned}
& x_{1}^{*}=2.25 \\
& x_{2}^{*}=3.75
\end{aligned}
$$



## Cutting plane: example

$$
\begin{aligned}
\max _{x} z=5 x_{1}+8 x_{2} & \\
\text { s.to } x_{1}+x_{2} & \leq 6 \\
5 x_{1}+9 x_{2} & \leq 45 \\
x_{1}, x_{2} \geq 0 & \in \mathbb{Z}
\end{aligned}
$$



$$
\begin{aligned}
& x_{1}^{*}=2.25 \\
& x_{2}^{*}=3.75
\end{aligned}
$$



## Cutting plane: example

$$
\begin{aligned}
& \max _{x} z= 5 x_{1}+8 x_{2} \\
& \text { s.to } x_{1}+x_{2} \leq 6 \\
& 5 x_{1}+9 x_{2} \leq 45 \\
& x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{aligned}
$$

## $2 x_{1}+3 x_{2} \leq 15$




## Cutting plane: example

$$
\begin{aligned}
\max _{x} z= & 5 x_{1}+8 x_{2} \\
\text { s.to } x_{1}+x_{2} & \leq 6 \\
5 x_{1}+9 x_{2} & \leq 45 \\
& x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{aligned}
$$

## $2 x_{1}+3 x_{2} \leq 15$




## Cutting plane: example

$$
\begin{aligned}
\max _{x} z= & 5 x_{1}+8 x_{2} \\
\text { s.to } x_{1}+x_{2} & \leq 6 \\
5 x_{1}+9 x_{2} & \leq 45 \\
x_{1}, x_{2} \geq 0 & \in \mathbb{Z}
\end{aligned}
$$

## $2 x_{1}+3 x_{2} \leq 15$



$$
\begin{aligned}
& x_{1}^{*}=3 \\
& x_{2}^{*}=3
\end{aligned}
$$



## Cutting plane: example

$$
\begin{aligned}
\max _{x} z= & 5 x_{1}+8 x_{2} \\
\text { s.to } x_{1}+x_{2} & \leq 6 \\
5 x_{1}+9 x_{2} & \leq 45 \\
& x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{aligned}
$$

## $2 x_{1}+3 x_{2} \leq 15$



$$
\begin{aligned}
& x_{1}^{*}=3 \\
& x_{2}^{*}=3
\end{aligned}
$$



## Cutting plane: example

$$
\begin{aligned}
& \max _{x} z= 5 x_{1}+8 x_{2} \\
& \text { s.to } x_{1}+x_{2} \leq 6 \\
& 5 x_{1}+9 x_{2} \leq 45 \\
& x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{aligned}
$$

## $2 x_{1}+3 x_{2} \leq 15$



$$
\begin{aligned}
& x_{1}^{*}=3 \\
& x_{2}^{*}=3
\end{aligned}
$$



## Cutting plane: example

$$
\begin{aligned}
& \max _{x} z= 5 x_{1}+8 x_{2} \\
& \text { s.to } x_{1}+x_{2} \leq 6 \\
& 5 x_{1}+9 x_{2} \leq 45 \\
& x_{1}, x_{2} \geq 0 \in \mathbb{Z}
\end{aligned}
$$

## $2 x_{1}+3 x_{2} \leq 15$



$$
\begin{array}{|l|}
\hline x_{1}^{*}=3 \\
x_{2}^{*}=3 \\
\hline
\end{array}
$$



## Example MILP Problem: Unit Commitment ${ }^{1}$

Decision on when to produce power and how much to produce in order to meet a demand forecast and minimize the associated costs over a planning horizon $T$.

Constraints

- Power balancing between production and demand
- Operation and capacity constraints.

Case Studies

- Simplest setting
- Quantized power-levels along with Minimum-up and down-time

[^0]
## Example MILP Problem: Optimization of Energy Grid

The plant is formed by:

- Two distributed generators
- $P_{\max , 1}=20[\mathrm{MW}]$
- $P_{\min , 1}=5[\mathrm{MW}]$
- $c_{1}=14[€ / \mathrm{MW}]$
- $O M_{1}=1[€]$
- $P_{\max , 2}=12[\mathrm{MW}]$
- $P_{\min , 2}=2[\mathrm{MW}]$
- $c_{2}=24[€ / \mathrm{MW}]$
- $O M_{2}=2[€]$
- A thermal energy storage
- $X_{\mathrm{s}, \min }=0[\mathrm{MWh}]$
- $X_{\mathrm{s}, \max }=50[\mathrm{MWh}]$
- $O M_{s}=1[€ / \mathrm{MW}]$


## Example MILP problem: optimization of energy grid

Microgrid composed by

- 1 energy storage unit
- 2 distributed generators (DG)


## Decision variables

## Continuous

- stored energy level $\left(X_{s}\right)$
- power level of the DG units ( $P_{i}$, with $i=1,2$ )
- power exchanged (positive for charging) with the storage unit $\left(P_{s}\right)$

Integer (binary)

- discharging(0)/charging(1) mode of the storage unit $\left(\delta_{s}\right)$
- off(0)/on(1) state of a DG unit ( $\delta_{i}$, with $i=1,2)$


## Example MILP problem: optimization of energy grid

The control strategy can be formulated as an optimization problem with the cost function representing microgrid running costs over the planning horizon.
minimize

$$
\sum_{k=1}^{T} c_{1} P_{1}(k)+c_{2} P_{2}(k)+O M_{1} \delta_{1}(k)+O M_{2} \delta_{2}(k)+O M_{s} P_{s}(k)
$$

subject to:

- power balancing for the electric loads $\quad P_{s}(k)=P_{1}(k)+P_{2}(k)-D(k)$
- energy storage dynamic (ideal case with unit efficiency) $X_{s}(k+1)=X_{s}(k)+P_{s}(k)$
- physical bounds $X_{s, \min } \leq X_{s}(k) \leq X_{s, \max }$
- operation constraints $\quad P_{\min , i} * \delta_{i}(k) \leq P_{i}(k) \leq P_{\max , i} * \delta_{i}(k) \quad i=1,2$.
$D$ output of a forecast service


## Example MILP Problem: optimization of energy grid

The control strategy is set with a sample time of 1 h and a planning horizon of 24 h . Figure shows the electricity power demand over the simulating horizon.


## Example MILP Problem: optimization of energy grid

Power level generated from the generator 1




Power exchanged with the storage



[^0]:    ${ }^{1}$ Link to YALMIP

